MATH 112A Review: Green's, Stokes', and Divergence Theorems

1. Use Green's theorem to evaluate

$$\int_C (-y^2\,dx + x\,dy),$$

where C is the unit square with positive orientation. Note that the unit square has corners (0,0), (0,1), (1,1), (1,0).

Solution: The region bounded by C is $R = [0,1] \times [0,1]$. By Green's theorem we have

$$\int_C (-y^2\,dx + x\,dy) = \iint_R (1+2y)dxdy = \int_0^1 \int_0^1 1dxdy + \int_0^1 \int_0^1 2ydxdy = 1 + 1 = 2.$$

2. Use the divergence theorem to evaluate

$$\iint_S F \cdot d\vec{S},$$

where $F(x,y,z) = (e^x, y\cos^2 x, z\sin^2 x - e^x z)$ and S is the surface of a cube with volume 2020.

Solution: We have $\nabla \cdot F = e^x + \cos^x + \sin^2 x - e^x = 1$. By divergence theorem, we have

$$\iint_S F \cdot d\vec{S} = \iiint_E 1dV,$$

where E is the cube enclosed by S. Since E has volume 2020, then

$$\iint_S F \cdot d\vec{S} = 2020.$$

3. Use Stoke's theorem to evaluate

$$\iint_S (\nabla \times F) \cdot d\vec{S},$$

where $F(x,y,z) = (0, xz, 0)$ and S is the upper half of the unit sphere: $x^2 + y^2 + z^2 = 1$ and $z \geq 0$.

Solution: The curve C that bounds S is the unit circle on the xy-plane. This follows by setting $z = 0$ in $x^2 + y^2 + z^2 = 1$. Using Stoke's theroem, we have that

$$\iint_S (\nabla \times F) \cdot d\vec{S} = \int_C F(\vec{r}) \cdot d\vec{r}.$$

Since we need the curve to be orientated in the positive direction, we set $\vec{r}(t) = (\cos t, \sin t, 0)$ and $t \in [0, 2\pi]$. Thus, $\vec{r}\,'(t) = (-\sin t, \cos t, 0)$ and $F(\vec{r}(t)) = (0, \cos t \sin t, 0)$. Therefore, $F(\vec{r}(t)) \cdot \vec{r}\,'(t) = \cos^2 t \sin t$. Hence,

$$\iint_S (\nabla \times F) \cdot d\vec{S} = \int_C F(\vec{r}) \cdot d\vec{r} = \int_0^{2\pi} \cos^2 t \sin t\,dt = -\cos^3(2\pi)/3 + \cos^3(0)/3 = 0.$$